Review of the Cohomology of Compact Lie Groups

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Abstract

We review the new computation of the cohomology of a compact Lie group by Mark Reeder [1]. Let G be a compact connected Lie group with a maximum torus T. $\mathfrak{g}$ and $\mathfrak{t}$ respectively are their Lie algebra respectively. Let W be the Weyl group. First we prove that as a W-module $H(G/T)$ is isomorphic to both the regular representation of W and the space $\mathscr{H}$ of W-harmonic polynomials on $\mathfrak{t}^*$. And $H(T)$ is naturally isomorphic to the exterior algebra of $\mathfrak{t}^*$. Then we construct a map: $\Psi : G/T \times T \to G$ with nonzero degree and find that $H(G)$ is isomorphic to $[H(G/T) \otimes H(T)]^W$ by Ψ^*. The latter space, which equals $[\mathscr{H} \otimes \Lambda\mathfrak{t}^*]^W$, is computed by Solomon's determination of the W-invariant differential forms on $\mathfrak{t}$ with polynomial coefficients.

0 Basic facts on compact Lie groups

Let G be a compact connected Lie group with a maximum torus T. T is abelian and its own centralizer in G. The Weyl group $W = N(T)/T$ is a finite group. Let $\mathfrak{g}$ and $\mathfrak{t}$ be the Lie algebras for G and T respectively. Because G is compact there is an inner product $<,>$ on $\mathfrak{g}$ which is invariant by the adjoint action $Ad(G)$. Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{g}$ with respect to this inner product, $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$. This implies $\mathfrak{m}$ is $Ad(T)$-invariant. The infinitesimal version of invariance of the inner product is

$$< [X,Y], Z > + < Y, [X,Z] >= 0, \ \forall \ X, Y, Z \in \mathfrak{g}$$

A regular element of $\mathfrak{t}$ is one whose $Ad(G)$-centralizer is G. We can find a regular element in $\mathfrak{t}$: since topologically T is S^l where $l = dimT$, there is an element t_0 of T whose powers form a dense set of T. Hence $C(t_0) = C(T) = T$. We choose an element H_0 in $\mathfrak{t}$ such that $expH_0 = t_0$. If $Ad(g)H_0 = H_0$ for some $g \in G$, then $Ad(g)expH_0 = exp(Ad(g)H_0) = t_0$ and $g \in \mathfrak{t}$.

The action $Ad(G)$ on $\mathfrak{g}$ induces a action of W on $\mathfrak{t}$. The injective map $i : \mathfrak{t} \to \mathfrak{g}$ induces a bijective map

$$\tilde{i} : \mathfrak{t}/Ad(W) \to \mathfrak{g}/Ad(G) \ .$$

T acts on $\mathfrak{m}$ via Ad. There is no element in $\mathfrak{m}$ which is invariant under $Ad(T)$, otherwise $\mathfrak{t}$ is not a maximum abelian subalgebra of $\mathfrak{g}$. Therefore because T is a torus, all the irreducible components of $\mathfrak{m}$ are real 2-dimensional and T acts on them by rotation: $\mathfrak{m} = \mathfrak{m}_1 \oplus ...\mathfrak{m}_v$. There are a finite set of linear functionals $\{\alpha_1, ..., \alpha_v\}$ such that for $H \in \mathfrak{t}$ the eigenvalues of $AdexpH$ on $\mathfrak{m}_i$ are $exp(\pm\sqrt{-1}\alpha_i(H))$. We choose a regular element $H_0 \in \mathfrak{t}$ and adjust the signs of α_i by requiring $\alpha_i(H_0) > 0$. Then we define the positive roots system $\Delta^+ = \{\alpha_1, ..., \alpha_v\}$. The action of W on $\mathfrak{t}$ is generated by the reflection (with invariant inner product) about the kernels of the positive roots. There are also a subset π of Δ^+ which is called the simple roots system and is a basis of $\mathfrak{t}^*$. The action of W on $\mathfrak{t}$ is actually generated by the reflection (with invariant inner product) about the kernels of the simple roots.

We use the notation of Chevalley's basis for Lie algebra. Let $\{X_i, X_{i+v}\}$ be an orthogonal basis for $\mathfrak{m}_i$ and with the basis:

$$ad(H)\mid_{\mathfrak{m}_i} = \begin{pmatrix} 0 & a_i(H) \\ -a_i(H) & 0 \end{pmatrix}$$

Hence for $1 \leq i, j \leq 2v, H \in \mathfrak{t}$, $< H, [X_i, X_j] >=< [H, X_i], X_j >$. From the matrix we know that it is nonzero only when $i - j = \pm v$. So that if $i - j \neq \pm v$, $[X_i, X_j]$ is perpendicular to $\mathfrak{m}_i$ and $[X_i, X_j] \in \mathfrak{m}$. Otherwise we define $H_i = [X_i, X_{i+v}]$. $Span\{H, X_i, X_{i+v}\}$ is Lie subalgebra which is isomorphic to $\mathfrak{s}u(2)$.

1 Invariant theory

Here we cite the invariant theory of the action of Weyl group from the book [2, chapter 2].

Let $\mathfrak{t}^*$ be the dual space of $\mathfrak{t}$. Weyl group acts on $\mathfrak{t}^*$ by contragredient representation:

$$(w\lambda)(H) = \lambda(Ad(w^{-1})H) \ \ \forall w \in W, \lambda \in \mathfrak{t}^*, H \in \mathfrak{t}$$

And let the graded algebras $\mathscr{S} = \oplus_{p=0}^{\infty}\mathscr{S}^p$ and $\Lambda = \oplus_{p=0}^{l}\Lambda^p$ be the symmetric and exterior algebras on $\mathfrak{t}^*$. ($l = dim\mathfrak{t}^*$) W acts on $\mathscr{S}$ and Λ naturally. Because as a ring $\mathscr{S}$ is isomorphic to the real polynomials ring in l variables, we will call the elements in $\mathscr{S}$ polynomials later.

Lemma 1.1. If $\mathfrak{g}$ is a simple Lie algebra, then each Λ^p $(0 \leq p \leq l)$ is an irreducible W-module.

Let $\mathscr{S}^W$ be the ring of invariant polynomials about the action of Wyel group on $\mathscr{P}$. Chevalley's theorem [2] gives the ring structure of $\mathscr{S}^W$.

Theorem 1.2. (Chevalley) *The ring $\mathscr{S}^W$ has algebraically independent homogenous generators $F_1, ..., F_l$, hence is a polynomial ring. If we number these generators so that $degF_1 \leq ...degF_2 \leq ...degF_l$ and let $m_i+1 = degF_i$, then $m_1+...+m_l = v$ and $(1+m_i)...(1+m_l) = |W|$.*

Remark 1.3. If $\mathfrak{g}$ is a simple Lie algebra, then $\mathfrak{t}^*$ is an irreducible W-module. Specially, there is no invariant element in $\mathfrak{t}^*$ and $\deg F_i > 1$.

Remark 1.4. We will see that $m_1, ...m_l$ determine the betti numbers of a compact connected Lie group. We have known the numbers $m_1, ...m_l$ for classical groups [3]. For a general compact connected Lie group, we know its Lie algebra is the direct sum of its center and simple Lie algebras and then this Lie group can be covered by the product of a central torus with a direct product of classical groups. Hence we can also get its $m_1, ...m_l$.

We determine the W-module structure of the polynomial ring $\mathscr{S}$. Let $\mathscr{D}$ be the ring of constant coefficient differential operators on $\mathscr{P}$. $\mathscr{D}$ is naturally isomorphic to the symmetric algebra $S(\mathfrak{t})$. Hence W acts on $\mathscr{D}$ and we defined $\mathscr{D}^W$ to be the W-invariant operators. Let $\mathscr{H}$ be the set of "harmonic polynomials" in $\mathscr{P}$:

$$\mathscr{H} = \{f \in \mathscr{S} : \mathscr{D}^W f = 0\}$$

Because $\mathscr{D}^W$ is a homogenous subring in $\mathscr{D}$, $f \in \mathscr{H}$ if and only if f is annihilated by all the homogenous elements in $\mathscr{D}^W$. Hence $\mathscr{H}$ is also a homogenous subring, and $\mathscr{H} = \oplus_{p=0}^{\infty} \mathscr{H}^p$. By the definition of the action W on $\mathfrak{t}^*$,

$$(wD)(wf) = w(Df), \quad \forall D \in \mathscr{D}, f \in \mathscr{S}, w \in W$$

This implies $\mathscr{H}$ is a W-module. Let $\mathscr{I}$ be the ideal in $\mathscr{S}$ generated by the elements of $\mathscr{S}^W$ of positive degree. $\mathscr{I}$ is also a W-module.

Theorem 1.5. *(1) Dim$\mathscr{H} = |W|$. (2) $\mathscr{S} = \mathscr{H} \oplus \mathscr{I}$. (3) The multiplication $\mathscr{H} \otimes \mathscr{S}^W \to \mathscr{S}$ is a linear isomorphism.*

Corollary 1.6. *As W-modules, $\mathscr{H} \simeq \mathscr{S}/\mathscr{I}$.*

In section 3, we will find that as W-module, $\mathscr{H}$ and $\mathscr{S}/\mathscr{I}$ are isomorphic to the regular representation of Weyl Group. [1]

Corollary 1.7. $\sum_{p \geq 0} \dim \mathscr{H}^p t^p = \prod_{i=1}^{l} (1 + t + t^2 + ... + t^{m_i})$.

Proof. From theorem 1.5 (3), we know that $(\oplus \mathscr{H}^p) \otimes (\oplus \mathscr{S}^{W,q}) \simeq (\oplus \mathscr{S}^s)$. Hence

$$(\sum_{p \geq 0} dim \mathscr{H}^p t^p)(\sum_{q \geq 0} dim \mathscr{S}^{W,q} t^q) = \sum_{s \geq 0} dim \mathscr{S}^s t^s$$

[1] We will use the fact later in this section to determine the dimension of $(\mathscr{S}/\mathscr{I} \otimes \Lambda)^W$, but of course we will use this dimension only after section 3.

It is easy to see that for a polynomial ring on one variable the generating function is $\frac{1}{1-t}$. Therefore $\mathscr{S}$, a polynomial ring on l variables, has the generating function $(\frac{1}{1-t})^l$. By theorem 1.2, $\mathscr{S}^W$, a polynomial ring on l polynomials with degrees $m_1+1, ..., m_l+1$, has the generating function $\prod_i(\frac{1}{1-t^{m_i+1}})$. Then we get $\sum_{p\geq 0}\dim \mathscr{H}^p t^p = \prod_{i=1}^{l}(1+t+t^2+...+t^{m_i})$ which shows that $dim\mathscr{H}^v = 1$ and $\mathscr{H}^v = 0$ for $p > v$.

Definition 1.8. Let V be an irreducible W-module. Suppose V is a constituent of $\mathscr{S}^b$, and not a constituent of $\mathscr{S}^c$, for any $c < b$. We call b the *birthday* of V.

Remark 1.9. For any $D \in \mathscr{D}^W$, D commutes with the action of W. Hence D is a homomorphism between W-modules. If b is the birthday of V, by Schur's lemma D must annihilate the respective constituent in $\mathscr{S}^c$ and then this constituent is in $\mathscr{H}^c$.

W acts on $\mathfrak{t}$ via adjoint action. Let $\varepsilon(w) = sign(detAd(w))$. ε forms a one-dimensional representation of W. We will find its birthday. We define the *primordial* harmonic polynomial to be

$$\Pi = \prod_{\alpha\in\Delta^+}\alpha \in \mathscr{S}^v.$$

Lemma 1.10. *(1)* Π *transforms by the sign character* ε *of* W. *(2)* v *is the birthday of* ε *and* $\mathscr{H}^v = span\{\Pi\}$.

Proof. Weyl group is generated by the refection about the simple roots hyperplanes. Let α_i be a simple root and r_i be the respective refection. $r_i[\Delta^+] = \Delta^+\backslash\{-\alpha_i\}$. Hence $r_i\Pi = -\Pi$. Any $w \in W$ is a product of r_i's, so $w\Pi = \varepsilon(w)\Pi$. Any polynomial transforming by ε must vanish on all roots hyperplanes, so it could be divisible by any root. Obviously any root is an irreducible elements in the ring $\mathscr{S}$, so the polynomial must be divisible by Π and has degree no less than Π. Therefore v is the birthday of ε and Π is harmonic. By corollary 1.7, $\mathscr{H}^v = span\{\Pi\}$.

Now consider the algebra $\mathscr{S}\otimes\Lambda$ of differential forms on $\mathfrak{t}$ with polynomial coefficients. The following theorem [2] describes the W-invariants in $\mathscr{S}\otimes\Lambda$.

Theorem 1.11. (Solomon) *The space* $(\mathscr{S}\otimes\Lambda)^W$ *of* W*-invariants in* $\mathscr{S}\otimes\Lambda$ *is a free* $\mathscr{S}^W$*-module with basis*

$$\{dF_{i_1}\wedge ... \wedge dF_{i_q} : 1 \leq i_1 < ... < i_q \leq l\}.$$

Lemma 1.12.[2] *Let* $F_1, ..., F_l$ *be* l *polynomials in a real polynomial ring in* l *variables.* $F_1, ..., F_l$ *are algebraically independent if and only if* $dF_1\wedge ... \wedge dF_l$ *is not zero.*

[2] I thank prof. Sjamaar's help on this lemma.

Proof. It is easy to see that if $dF_1 \wedge ... \wedge dF_l$ is nonzero then $F_1, ..., F_l$ are algebraically independent. Consider $y_1 = F_1(x_1, ...x_l), ..., y_l = F_l(x_1, ...x_l)$ as a map from $\mathbb{R}^l$ to $\mathbb{R}^l$. Suppose there is a polynomial $Q(y_1, ...y_l)$ such that $Q(F_1, ..., F_l) = 0$, i.e., Q's zero set contains the image of the map. But $dF_1 \wedge ... \wedge dF_l$ is nonzero, there must be a point with nondegenerate Jacobian. By inverse function theorem, the image contains a open set. Then the fact that $Q(y_1, ...y_l)$ vanishes on a open set implies $Q \equiv 0$.

The converse case is harder to proof. Here we use the proof from [2, Chapter III]. We know the $\mathbb{R}[x_1, ..., x_l]$ has transcendency degree l, so any $l+1$ polynomials are algebraically dependent. In particular, $x_i, F_1, ..., F_l$ are algebraically dependent. We choose a polynomial $Q_i(x_i, z_1, ..., z_l)$ of minimal degeree $e_i > 0$ in x_i such that $Q_i(x_i, F_1, ..., F_l) = 0$. Applying $\partial/\partial x_k$, we obtain

$$\sum_{r=1}^{l} \frac{\partial Q_i}{\partial z_r}(x_i, F_1, ..., F_l)\frac{\partial F_r}{\partial x_k} + \delta_{ik}\frac{\partial Q_i}{\partial x_k}(x_i, F_1, ..., F_l) = 0$$

Let

$$A_{ir} = \frac{\partial Q_i}{\partial z_r}(x_i, j_1, ...j_r),\ B_{rk} = \frac{\partial F_r}{\partial x_k},\ C_{ij} = \delta_{ik}\frac{\partial Q_i}{\partial x_i}(x_i, j_1, ...j_r)$$

We get the matrix identity (We can regard elements of matrix as elements in the quotient field of polynomial ring)

$$AB = C$$

If $detC = 0$, then there is a i such that

$$\frac{\partial Q_i}{\partial x_i}(x_i, j_1, ...j_r) = 0.$$

This is a contradiction to our assumption of minimal degree. Hence $detB \neq 0$ and $dF_1 \wedge ... \wedge dF_l$ is nonzero.

Proof of theorem 1.11. Let $x_1, ..., x_l$ be a basis of $\mathbf{t}^*$.

$$dF_{i_1} \wedge ... \wedge dF_{i_l} = J dx_1 \wedge ... \wedge dx_l.$$

By lemma 1.12., the Jacobian J is a nonzero polynomial of degree $m_1 + ... + m_l = v$. The left side is W-invariant and $dx_1 \wedge ... \wedge dx_l$ transforms by the sign character ε. Hence J must also transforms by ε and because it has degree v J is a nonzero multiple of the primordial harmonic polynomials Π.

$$dF_{i_1} \wedge ... \wedge dF_{i_l} = c\Pi dx_1 \wedge ... \wedge dx_l,\ c \neq 0$$

For a sequence $I = i_1 < ... < i_q$, define its complement to be I', the increasing sequence of all integers in $\{1, ...l\} - \{i_1, ..., i_q\}$. Let $dF_I = dF_{i_1} \wedge ... \wedge dF_{i_q}$ and k be the quotient field of $\mathscr{S}$.

If $\sum_I f_I dF_I = 0$, after multiplying by $dF_{S'}$, we get $\sum_I f_I dF_I \wedge dF_{S'} = 0$. All the I's which are not S but have the length of S are killed. Because the component which has degree l about exterior product $\pm cg_S \pi dx_1...dx_l$ should also be zero, we get $f_S = 0$. Therefore dF_I's are k-independent.$dim_k(k \otimes \Lambda) = dim_{\mathbb{R}}\Lambda = 2^l$ and there are 2^l different I's, so dF_I's form a k-basis of $k \otimes \Lambda$ and are in particular linearly independent over $\mathscr{S}^W$. Now suppose $\omega \in \mathscr{S} \otimes \Lambda$ is homogenous about the degree of exterior product and W-invariant. Let $\omega = \sum_I g_I dF_I$, $g_I \in k$. Multiplying by $dF_{S'}$ again, we have

$$\omega \wedge dF_{S'} = \pm cg_S \pi dx_1...dx_l \ \in [\mathscr{S} \otimes \Lambda]^W$$

Hence $g_S \Pi$ should be a polynomial in $\mathscr{S}$ and transforms by ε. So in $\mathscr{S}$, $\Pi | g_S \Pi$. This implies g_S should be a polynomial in $\mathscr{S}$ and W-invariant. Therefore the space $(\mathscr{S} \otimes \Lambda)^W$ is a free $\mathscr{S}^W$-module with basis dF_I.

We need work a little more on the structure of $(\mathscr{S}/\mathscr{I} \otimes \Lambda)^W$.

Corollary 1.13. *For $\omega \in (\mathscr{S} \otimes \Lambda)$, Let $\omega' \in (\mathscr{S}/\mathscr{I} \otimes \Lambda)$ be the different form with coefficients of ω module $\mathscr{I}$. Then $\{dF'_{i_1} \wedge ... \wedge dF'_{i_q} : 1 \le i_1 < ... < i_q \le l\}$ is a basis of $(\mathscr{S}/\mathscr{I} \otimes \Lambda)^W$.*

Proof. We have an exact sequence

$$0 \to (\mathscr{I} \otimes \Lambda)^W \to (\mathscr{S} \otimes \Lambda)^W \overset{\omega \mapsto \omega'}{\to} (\mathscr{S}/\mathscr{I} \otimes \Lambda) \to 0$$

Form Solomon's theorem $\{dF'_{i_1} \wedge ... \wedge dF'_{i_q} : 1 \le i_1 < ... < i_q \le l\}$ spans $(\mathscr{S}/\mathscr{I} \otimes \Lambda)^W$ with coefficients in $\mathbb{R}$. (All W-invariant polynomials with degree larger than zero are in $\mathscr{I}$.) To prove it is a basis, we use the fact $\mathscr{S}/\mathscr{I}$ affords the regular representation and the following lemma to find the real dimension of $(\mathscr{S}/\mathscr{I} \otimes \Lambda)^W$:

Lemma 1.14. *Let W be a finite group. Let (W, V) be the regular representation of W and (W, U) be any representation. Then $dim(V \otimes U)^W = dimU$.*

Proof. The character of regular representation $\chi_V(e) = |W|$, $\chi_V(w) = 0, w \ne e$. So $\chi_{U \otimes V}(e) = |W| dimU$, $\chi_{U \otimes V}(w) = 0, w \ne e$. It implies $dim(V \otimes U)^W = dimU$.

Therefore $dim_{\mathbb{R}}(\mathscr{S}/\mathscr{I} \otimes \Lambda)^W = 2^l$ and corollary 1.13 holds. We have the following

Corollary 1.15. *(1) $(\mathscr{S}/\mathscr{I} \otimes \Lambda)^W$ is an exterior algebra with generators $dF'_i \in (\mathscr{S}/\mathscr{I}^{m_i} \otimes \Lambda^1)^W$, $1 \le i \le l$. (2) Multiplicity formula*

$$\sum_{n=0}^{v} dimHom_W(\Lambda^q, \mathscr{H}^n) u^n = s_q(u^{m_1}, ...u^{m_l})$$

Proof. (1) It follows corollary 1.13. (2) As W-modules, $Hom_W(\Lambda^q, \mathscr{H}^n) \simeq [(\Lambda^q)^* \otimes \mathscr{H}^n]^W$. Any real representation is isomorphic to its contragredient representation, therefore $dimHom_W(\Lambda^q, \mathscr{H}^n) = dim[\mathscr{H}^n \otimes \Lambda^q]^W$. Because $(\mathscr{S}/\mathscr{I})^n \simeq \mathscr{H}^n$ and from (1), the dimension of $dim[\mathscr{H}^n \otimes \Lambda^q]^W$ is the number of different products of generators with total degree of polynomial n.

Remark 1.16. In particular, the birthday of Λ^q is $m_1 + ... + m_q$, if $\mathfrak{g}$ is simple.

2 Invariant differential forms

Let G be a compact connected Lie Group which acts transitively on a manifold M. This implies M is also compact. Let τ_g be the diffeomorphism of M corresponding to $g \in G$.

Lemma 2.1. *τ_g^* acts trivially on the cohomology of M.*

Proof. Because G is compact and connected, for any $g \in G$ there is a $X \in \mathfrak{g}$ such $expX = g$. Let

$$A_m : \ G \to M, \ g \mapsto gm$$

Define $\tilde{X}|_m = (A_m)_*(X)$ so that $\tilde{X}$ is a smooth vector field on M. For any closed form ω

$$\frac{d}{dt}|_{t=t_0} \tau^*_{exptX}\omega = L_{\tilde{X}} \tau^*_{expt_0X}\omega = i(\tilde{X})\tau^*_{expt_0X} d\omega + d \circ i(\tilde{X})\tau^*_{expt_0X}\omega = d \circ i(\tilde{X})\tau^*_{expt_0X}\omega$$

Hence

$$\tau_g^*\omega - \omega = d\int_0^1 i(\tilde{X})\tau^*_{exptX}\omega dt$$

τ_g^* does't change the cohomology class of ω.

Definition 2.2. An *invariant differential form* of M is a differential form ω on M such that $\tau_g^*\omega = \omega$ for all $g \in G$. $\Omega(M)^G$ is the set of invariant differential forms.

Lemma 2.3. For any closed form ω on M, there is an invariant form ω' which is in the same cohomology class of ω.

Proof. Because G is compact, there is an invariant integration on G. Let $\omega' = \int_G \tau_g^*\omega$. Then

$$\tau_h^*\omega' = \int_G \tau^*_{gh}\omega = \int_G \tau_g^*\omega = \omega'$$

The exterior derivative d commutes with τ_g^*, so we have the subcochain $\{H(M)^G, d\}$ of invariant forms on M. There is a natural mapping $i : H(M)^G \to H(M)$. Lemma 2.3 shows that i is surjective. If w is an invariant exact form, say, $w = d\alpha$, then $w = d\int_G \tau_g^* a$. Therefore i is also injective and we have the following lemma

Lemma 2.3. *As cochains,* $\{H(M)^G, d\} \simeq \{H(M), d\}$.

We will use the Lie algebra of G to compute this cochain. Choose a point $o \in M$, and let $K \subset G$ be its stabilizer. Since G is compact, G/K is a homogeneous manifold which is isomorphic to M. We have an orthogonal decomposition $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{n}$, where $\mathfrak{r}$ is the Lie algebra of K. $Ad(K)$ acts on $\mathfrak{r}$ trivially so that this decomposition is kept by $Ad(K)$. $\mathfrak{n}$ is isomorphic to the tangent space $T_o(M)$. Let $(\Lambda^p\mathfrak{n}^*)^K$ be the subspace of elements in $\Lambda^p\mathfrak{n}^*$ invariant under $Ad(K)$.

Lemma 2.4. *As linear spaces,* $\Omega^p(M)^G \simeq (\Lambda^p\mathfrak{n}^*)^K$. *Furthermore,* $\Omega(M)^G \simeq (\Lambda\mathfrak{n}^*)^K$ *as rings.*

Proof. $(A_o)_*$ is an isomorphism between $\mathfrak{n}$ and $T_o(M)$ which is also a equivalence of K's actions via $Ad(K)$ and $(\tau_K)_*$. The mapping induces the isomorphism between $(\Lambda^p\mathfrak{n}^*)^K$ and $(\Lambda^p(T_o^*M))^K$. We claim that the later space is linearly isomorphic to $\Omega^p(M)^G$. Let $l : \Omega^p(M)^G \to \Lambda^p(T_o^*M)^K$ be the mapping:

$$\omega \mapsto \omega|_o, \ \omega \in \Omega^p(M)^G$$

It is well-defined because $(\tau_k^*)\omega|_o = \omega|_o$. It's also injective because if $\omega|_o$ is zero by invariance and the fact G acts on M transitively ω is zero everywhere. To show it is surjective we define a differential form $\tilde{\alpha}$ on M for any element $\alpha \in \Omega^p(M)^G$: $\tilde{\alpha}|_{go} = (\tau_{g^{-1}}^*)\alpha|_o$. It is well defined because if $g_1 o = go$ then $g_1^{-1}g \in K$ and $(\tau_{g_1^{-1}}^*)\alpha|_o = (\tau_{g^{-1}}^*)(\tau_{g_1^{-1}g}^*)\alpha|_o = (\tau_{g^{-1}}^*)\alpha|_o$. $\tilde{\alpha}$ is also G-invariant, $(\tau_g^*\tilde{\alpha})|_m = (\tau_g^*)\tilde{\alpha}|_{gm} = (\tau_g^*)(\tau_{h^{-1}g^{-1}}^*)\alpha|_o = (\tau_{h^{-1}}^*)\alpha|_o = \alpha|_m$, for any $g \in G$, $m \in M$ and assuming $ho = m$. Obviously $l(\tilde{\alpha}) = \alpha$. The other part of the lemma comes from the fact that the map $(A_o)^*$ commutes with exterior product.

By this lemma, we have a derivative δ on $(\Lambda^p\mathfrak{n}^*)^K$ to make this diagram commutes.

$$\begin{array}{ccc} (\Lambda^p\mathfrak{n}^*)^K & \longrightarrow & \Omega^p(M)^G \\ \delta\downarrow & & d\downarrow \\ (\Lambda^{p+1}\mathfrak{n}^*)^K & \longrightarrow & \Omega^{p+1}(M)^G \end{array}$$

Proposition 2.5. *δ is determined by* [3]

$$\delta\omega(X_0, ..., X_p) = \sum_{i<j}(-1)^{i+j}\omega([X_i, X_j]_n, X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_p)$$

$\omega \in (\Lambda^p\mathfrak{n}^*)^K$, $X_0, ...X_p \in \mathfrak{n}$ and $[X_i, X_j]_n$ is the projection of $[X_i, X_j]$ into $\mathfrak{n}$ along $\mathfrak{r}$. Then $\{(\Lambda^p\mathfrak{n}^*)^K, \delta\} \simeq \{H(M)^G, d\} \simeq \{H(M), d\}$.

Proof. Let $\tilde{\omega}$ be ω's image in $\Omega^p(M)^G$ as we assumed before. For $X_0, ...X_p \in \mathfrak{n}$, define $\tilde{X}_i$ to be vector fields on M by $\tilde{X}_i|_m = (A_m)_*X_i$.

$$d\tilde{\omega}(\tilde{X}_0, ..., \tilde{X}_p) = \sum_{i=0}^{p}(-1)^i\tilde{X}_i\tilde{\omega}(\tilde{X}_0, ..., \hat{\tilde{X}}_i, ..., \tilde{X}_p) + \sum_{i<j}(-1)^{i+j}\tilde{\omega}([\tilde{X}_i, \tilde{X}_j], \tilde{X}_0, ..., \hat{\tilde{X}}_i, ..., \hat{\tilde{X}}_j, ..., \tilde{X}_p)$$

Because $\tilde{\omega}$ is an invariant form, $L_{X_i}\tilde{\omega} = 0$.

$$\begin{aligned}\tilde{X}_i\tilde{\omega}(\tilde{X}_0, ..., \hat{\tilde{X}}_i, ..., \tilde{X}_p) &= L_{\tilde{X}_i}(\tilde{\omega}(\tilde{X}_0, ..., \hat{\tilde{X}}_i, ..., \tilde{X}_p) \\ &= \sum_{j=0,\ j\neq i}^{p} \tilde{\omega}(\tilde{X}_0, ..., \hat{\tilde{X}}_i, ..., [\tilde{X}_i, \tilde{X}_j], ..., \tilde{X}_p)\end{aligned}$$

Sum them and we get

$$d\tilde{\omega}(\tilde{X}_0, ..., \tilde{X}_p) = -\sum_{i<j}(-1)^{i+j}\tilde{\omega}([\tilde{X}_i, \tilde{X}_j], \tilde{X}_0, ..., \hat{\tilde{X}}_i, ..., \hat{\tilde{X}}_j, ..., \tilde{X}_p)$$

Let Y_i be the respective *right* invariant vector field on G of X_i. It is easy to see $(A_o)_*Y_i = \tilde{X}_i$ and $(A_o)_*([Y_i, Y_j]) = [(A_o)_*Y_i, (A_o)_*Y_j]$. In particular, $[\tilde{X}_i, \tilde{X}_j]|_o$=$(A_o)_*[Y_i, Y_j]|_e$. By the convention of the definition of Lie algebra, $[Y_i, Y_j]|_e = -[X_i, X_j]$. Hence we get $[\tilde{X}_i, \tilde{X}_j]|_o = -(A_o)_*[X_i, X_j] = -(A_o)_*[X_i, X_j]_n$. It implies

$$\delta\omega(X_0, ..., X_p) = \sum_{i<j}(-1)^{i+j}\omega([X_i, X_j]_n, X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_p).$$

Remark 2.6. Assume that K is connected. Let $n = dim\mathfrak{n}$. Then $\Lambda^p\mathfrak{n}^*$ is one-dimensional and K acts on it by multiplying the determinant of $Ad(k^{-1}), k \in K$. But K is also compact,

[3]This formula is different from that in [1] where there is an factor $1/(p + 1)$. It comes from the different definition of exterior product.

so the determinant is a homomorphism of K to a compact connected subgroup of $\mathbb{R}^*$, i.e. $\{1\}$. Therefore K acts on $\Lambda^p\mathfrak{n}^*$ trivially and $dim(\Lambda^p\mathfrak{n}^*)^K = 1$. So there is nonzero invariant form of degree of n on M, and it is nonzero everywhere. It implies that M is orientable. In particular, G/T is orientiable.

We consider the special case that G acts on G. Now $\mathfrak{r} = 0$, $\mathfrak{n} = \mathfrak{g}$. Because G acts on itself by both left and right action, we have a bi-invariant representive in every cohomology class by averaging:

$$\int_G \int_G (L_g^*)(R_h^*)\omega dgdh$$

We denote the set of bi-invariant forms $\Omega(G)^{bi}$. Similarly we have

Lemma 2.7. As cochains, $\{\Omega(G)^{bi}, d\} \simeq \{\Omega(G), d\}$

The value of a bi-invariant form at e must be $Ad(G)$-invariant, and similarly we have,

Lemma 2.8. *As linear spaces, $(\Omega^p(G))^{bi} \simeq (\Lambda^p\mathfrak{g}^*)^G$. Furthermore, $(\Omega(G))^{bi} \simeq (\Lambda\mathfrak{g}^*)^G$ as rings.*

We can also define the derivative δ on $\Lambda\mathfrak{g}^*$ to make it a cochain which is isomorphic to $\{\Omega(G)^{bi}, d\}$. The explicit formula of δ for has been given by proposition 2.5.

Proposition 2.8. *$\delta = 0$ for $\Lambda\mathfrak{g}$ and $H^p(G) \simeq (\Lambda^p\mathfrak{g}^*)^G$.*

Proof. For any $\omega \in (\Lambda^p\mathfrak{g}^*)^G$, $X, X_1, ..., X_p \in \mathfrak{g}$

$$\omega(X_1, ...X_p) = (Ad(exp(-tX))\omega)(X_1, ...X_p) = \omega(Ad(exp(tX)X_1, ...Ad(exp(tX)X_p)$$

or taking its derivative

$$\omega([X, X_1], ...X_p) + ... + \omega(X_1, ..., [X, X_p]) = 0$$

Sum several such identities with suitable coefficients

$$\sum_{j=0}^{i-1}(-1)^{j+1}\omega([X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_p) = 0$$

and

$$\sum_{j=i+1}^{p}(-1)^{j}\omega([X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_p) = 0$$

Then multiply by $(-1)^i$ and sum over i,

$$\delta\omega = 0$$

This proposition implies $H((\Lambda\mathfrak{g}^*)^G) = (\Lambda\mathfrak{g}^*)^G$ and hence $H^p(G) \simeq (\Lambda^p\mathfrak{g}^*)^G$.

Therefore it is easy to compute $H(T)$, the cohomology of maximum tours. T is an Abelian Lie group, $Ad(T)$ acts on T trivially. We have

Theorem 2.9. *As a W-module, $H(T)$ is algebraically isomorphic to $\Lambda\mathfrak{t}^*$.*

3 The cohomology of flag manifold

We use Morse theory [4] to compute the cohomology of the flag manifold G/T as linear spaces. Then by the character of representation of Weyl group we compute the cohomology of G/T as W-module. Finally we compute the cohomology of G/T as algebra with Borel's theorem[5].

Recall that a real valued function f on a differential manifold is call Morse Function if and only if it has nondegenerate Hessian at every critical point ($df|_x = 0$).

Choose a regular elements $H_0 \in \mathfrak{t}$ and define positive roots about H_0 . Define a function $f : G/T \to R$ by

$$f(gT) =< Ad(g)H_0, H_0 >$$

It is obviously well-defined. Now we find the critical points of f. G acts on G/T, and we define $A_{hT} : G \to G/T$ by $A(g) = ghT$. For $X \in \mathfrak{g}$, let $\tilde{X}$ be a vector field on G/T such that $\tilde{X}|_{gT} = (A_{gT})_*X$. Because G is a fibre bundle on base space G/T with fibre T, $(A_{gT})_*$ is surjective for any gT.

Lemma 3.1. *The set of critical points of f in G/T is W, the Weyl group of G.*

Proof.

$$\begin{aligned}\tilde{X}f(gT) &= \frac{d}{ds}|_{s=0} < Ad(exp(sX)g)H_0, H_0 > \\ &= \frac{d}{ds}|_{s=0} < Ad(g)H_0, Ad(exp(-sX))H_0 > \\ &=< Ad(g)H_0, [H_0, X] >\end{aligned}$$

Since the centralizer of H_0 in $\mathfrak{g}$ is exactly $\mathfrak{t}$, $ad(H_0)$: $\mathfrak{m} \to \mathfrak{m}$ is a bijective. So gT is a critical point of f if and only if $< Ad(g)H_0, \mathfrak{m} >= 0$. This means $Ad(g)H_0 \in \mathfrak{t}$. Recall the bijective:

$$\tilde{i} : \mathfrak{t}/Ad(W) \to \mathfrak{g}/Ad(G)\ .$$

Hence there is a $wT \in W \subset G/T$ such that $Ad(w)H_0 = Ad(g)H_0$. Then $gw^{-1} \in T \subset N(T)$ and $gT \in W$. So the critical points of f are wT, for $w \in W$.

Let $X_1, X_2, ..., X_{2v}$ be the orthogonal basis of $\mathfrak{m}$ in the section 0. Because $(A_{wT})_* H = d/ds|_{s=0} exp(sH)wT = d/ds|_{s=0} w exp(sAd(w^{-1})H)T = d/ds|_{s=0} wT = 0$, $\forall H \in \mathfrak{t}$. The values of $\tilde{X}_i$ at wT form a basis for the tangent space. We can compute the Hessian of f at wt about this basis.

Lemma 3.2. *f is a Morse function on G/T*

Proof.

$$\begin{aligned} h_{ij}(wT) &= \tilde{X}_i \tilde{X}_j f(wT) \\ &= \frac{d}{ds}|_{s=0} < Ad(exp(sX_i)w)H_0, [H_0, X_j] > \\ &= < [X_i, Ad(w)H_0], [H_0, X_j] > \end{aligned}$$

By Chevalley's basis, $h_{ij} = 0$ if $i \neq j$. Otherwise,

$$h_{ii}(w) = -\alpha_i(Ad(w)H_0)\alpha_i(H_0)$$

$Ad(w)H_0$ is also an regular element in $\mathfrak{t}$, so the Hessian is nonsingular. Because $\alpha_i(Ad(w)H_0) = Ad(w^{-1}\alpha_i)H_0$, the number of negative eigenvalues equals twice the number $m(w)$ of positive roots α such that $w^{-1}\alpha$ is again positive.

By the main theorem of Morse Theory, we get

Theorem 3.3. *The Poincare polynomial of G/T is $\sum_{w \in W} u^{2m(w)}$.*

Proof. The morse function f on G/T shows G/T has the homotopy type of a CW-Complex whose cells are all even-dimensional. Then $dim H^n(G/T) =$ number of cells of dimension n, if n is even. Otherwise $dim H^n(G/T) = 0$. It also implies that $H(G/T)$ is a commutative ring.

Corollary 3.4. $H^{odd}(G/T) = 0$ and $H^{odd}(G/T) = dim(G/T) = \chi(G/T) = |W|$.

We define an action of W on G/T,

$$w(gT) = gw^{-1}T, \ w \in W, \ gT \in G/T$$

It's well-defined, because if we choose other representatives of w and gT, say wt_1 and gt_2T, then $wt_1(gt_2T) = gt_2t_1^{-1}w^{-1}T = gw^{-1}Ad(w)(t_2t_1^{-1})T = gw^{-1}T$. We also define the action of Weyl Group on $H(G/T)$,

$$w(\omega) = (w^{-1})^*\omega, \ w \in W, \omega \in H(G/T)$$

It is easy to see that it is a left action.

Theorem 3.5. *As a W-module, $H(G/T)$ is isomorphic to the regular representation of W*

Proof. The Lefschetz number of the mapping w on G/T is,

$$\sum_{i=0}^{dimG/T}(-1)^i Trace(w^*|_{H^i(G/T)}) = \sum_{i\ even} Trace(w^*|_{H^i(G/T)}) = \chi_{H(G/T)}(w^{-1})$$

If $w \neq e$, there is no fixing point on G/T. By Lefschetz fixing point theorem, the Lefschetz number and hence the character of representation G(H/T) are zero if $w \neq e$. If $w = e$, $\chi_{H(G/T)}(w^{-1}) = dimH(G/T)$ obviously. So $H(G/T)$ is isomorphic to the regular representation of W.

We also want to determine the structure of $H(G/T)$ as a ring. Recall the graded ring $\mathscr{S}$ of polynomials in $\mathfrak{t}$ and its ideal $\mathscr{I}$ generated by the W-invariant polynomials of positive degrees.

Theorem 3.6. (Borel) *There is a degree-doubling W-equivalent ring isomorphism*

$$\tilde{c} : \mathscr{S}/\mathscr{I} \to H(G/T)$$

In the other words, $\mathscr{H}_{(2)} \simeq H(G/T)$, where $\mathscr{H}_{(2)}$ is $\mathscr{H}$ with the grading degrees doubled.

Proof. We define the mapping c as: first for any $\lambda \in \mathfrak{t}^*$, extended to a functional on $\mathfrak{g}$ by making it zero on $\mathfrak{m}$, define a two-form ω_λ on $\mathfrak{m}$ by

$$\omega_\lambda(X, Y) = \lambda([X, Y])$$

$Ad(t)(\omega_\lambda)(X, Y) = \lambda([Ad(t^{-1})X, Ad(t^{-1})Y]) = \lambda(Ad(t^{-1})([X, Y])) = \lambda([X, Y])$, Hence ω_λ is an $Ad(T)$-invariant form. By the discussion in section 2, we know that it corresponds to a G left-invariant form $\tilde{\omega}_\lambda$. Now we have a mapping from $\mathfrak{t}^*$ to $H^2(G/T)$ and we should check that it is W-equivalent. For any $w \in W$, we choose a representative w_1 in G.

$$((w^{-1})^*\tilde{\omega}_\lambda)|_{eT} = (w^{-1})^*(\tilde{\omega}_\lambda|_{wT}) = (w^{-1})^*(\tau_{w_1^{-1}})^*(\tilde{\omega}_\lambda|_{eT})$$

We have a commutative diagram,

$$\begin{array}{ccc} G & \xrightarrow{\pi} & G/T \\ {\scriptstyle Ad(w_1^{-1})}\downarrow & & \downarrow {\scriptstyle \tau_{w_1^{-1}} \circ w^{-1}} \\ G & \xrightarrow{\pi} & G/T \end{array}$$

Therefore $((w^{-1})^*\tilde{\omega}_\lambda)|_{eT} = \tilde{\omega}_{Ad(w_1^{-1})^*\lambda}|_{eT} = \tilde{\omega}_{w\lambda}|_{eT}$. Because the actions of G on G/T and W on G/T are commutative, we know that $(w^{-1})^*(\tilde{\omega}_\lambda)$ is also left-invariant. So $(w^{-1})^*\tilde{\omega}_\lambda = \tilde{\omega}_{w\lambda}$.[4] This verifies that $\lambda \to \tilde{\omega}_\lambda$ is W-equivalent. Define $c(\lambda) = [\tilde{\omega}_\lambda] \in H^2(G/T)$ and it is also a W-equivalent mapping. This extends to a W-equivalent degree-doubling ring homomorphism

$$c : \mathscr{S} \to H(G/T)$$

Since $H(G/T)$ is the regular representation of W, its W-invariants are one-dimensional and therefore $H^0(G/T)$. So any $f \in \mathscr{S}^W$ with nonzero degree is killed by c and $\mathscr{I}$ is in the kernel of c. We have a mapping

$$\tilde{c} : \mathscr{S}/\mathscr{I} \to H(G/T)$$

In section 1, we have known that $dim\mathscr{S}/\mathscr{I} = |W|$. In order to prove that it is an isomorphism we only need to prove it is injective. Because $\mathscr{S} = \mathscr{H} \oplus \mathscr{I}$, we will prove the theorem by showing that the restriction of c to $\mathscr{H}$ is injective.

First we start in the top dimension. $Dim\mathscr{H}^v = 1$ and $\mathscr{H}^{2v} = span\{\Pi\}$ where $\Pi = \prod_{\alpha\in\Delta^+} \alpha$ is the primordial harmonic polynomial. We evaluate $\tilde{c}(\Pi)$ explicitly. For any positive root α_i, write ω_i for ω_{α_i}. If $\omega_1 \wedge ... \wedge \omega_v$ is not zero, then the invariant form $\tilde{\omega}_1 \wedge ... \wedge \tilde{\omega}_v$ is nonzero at any point of G/T and hence of nontrivial cohomology class because $dimG/T = 2v$. Hence $\tilde{c}(\Pi) = [\tilde{\omega}_1 \wedge ... \wedge \tilde{\omega}_v] \neq 0$. So we only need to show that $\omega_1 \wedge ... \wedge \omega_v \neq 0$. We use X_i's as the basis of $\mathfrak{m}$.

$$\begin{aligned}
&\omega_1 \wedge ... \wedge \omega_v(X_1, X_{1+v}, ..., X_v, X_{2v}) \\
= &\sum_{\sigma\in S_{2v}} sgn(\sigma)\omega_1(X_{\sigma(1)}, X_{\sigma(1+v)})...\omega_v(X_{\sigma(v)}, X_{\sigma(2v)}) \\
= &\sum_{\sigma\in S_{2v}} sgn(\sigma)\alpha_1([X_{\sigma(1)}, X_{\sigma(1+v)}])...\alpha_v([X_{\sigma(v)}, X_{\sigma(2v)}])
\end{aligned}$$

Now $\alpha_i([X_{\sigma(i)}, X_{\sigma(i+v)}]) = 0$ unless $[X_{\sigma(i)}, X_{\sigma(i+v)}] \in \mathfrak{t}$. So the σ^{th} term is nonzero only if σ permutes the pair $\pi_i = i, i+v$. $Sgn(\sigma)$ cancels the sign induced by switching members in each pair. Hence

$$\begin{aligned}
&\omega_1 \wedge ... \wedge \omega_v(X_1, X_{1+v}, ..., X_v, X_{2v}) \\
= &2^v \sum_{\sigma\in S_v} \alpha_1(X_{\sigma(1)}, X_{v+\sigma(1)})...\alpha_v(X_{\sigma(v)}, X_{v+\sigma(v)}) \\
= &2^v \sum_{\sigma\in S_v} \alpha_1(H_{\sigma(1)})...\alpha_1(H_{\sigma(1)}) \\
= &2^v \partial_1 ... \partial_v \Pi
\end{aligned}$$

[4] It is different from the formula in [1].

Here $\partial_i = H_i \in (\mathfrak{t}^*)^* = \mathfrak{t}$. We have a perfect pairing

$$\mathscr{D}^v \otimes \mathscr{S}^v \to \mathbb{R}$$

and by the definition of the action of W on $\mathfrak{t}$ and $\mathfrak{t}^*$ the pairing is W-invariant. There must exist polynomials in $\mathscr{D}^v$ such that pair nontrivially with Π. But any irreducible W-module can only pair nontrivially with its dual, so these polynomial which pair nontrivially with Π must also transforms as the sign representation of Weyl group. Because $\mathscr{S}^v \simeq (\mathscr{D}^v)^*$ as W-modules there is only one copy of the sign representation in $\mathscr{D}^v$, which is afford by $\partial_1 ... \partial_v$. Therefore $\partial_1 ... \partial_v \Pi \neq 0$. This complete the proof that $\tilde{c}(\Pi) \neq 0$.

We now inductively assume that $\tilde{c} : \mathscr{H}^k \to H^{2k}(G/T)$ is injective for $k \leq v$. Let $V = \mathscr{H}^{k-1} \cap ker\tilde{c}$. Both $\mathscr{H}^{k-1}$ and $ker\tilde{c}$ are W-invariant, so does V. The sign representation does not occur in $\mathscr{H}^{k-1}$, so there is positive root α whose corresponding refection s_α in Weyl Group does not act by $-I$ on V. The eigenvalues of the action of s_α on V can only be 1 or -1. Decompose $V = V_+ \oplus V_-$ according to the eigenspaces of s_α. So if $V \neq 0$ then $V_+ \neq 0$, so take $f \in V_+$. Now $\tilde{c}(\alpha f) = \tilde{c}(\alpha)\tilde{c}(f) = 0$, and αf is in degree k, so we must have $\alpha f \in \mathscr{I}$ by the induction hypothesis. Choose a basis $\{h_1, ..., h_{|W|}\}$ in $\mathscr{H}$ such that $h_1, ..., h_r$ s_α-skew and the rest s_α-invariant. By theorem 1.5, we can write $\alpha f = \sum h_i \sigma_i$ with σ_i W-invariant. Here σ_i should have positive degrees because suppose we have $\alpha f = \sum h_j \sigma_j + h$ with $deg\sigma_j > 0$ and $h \in \mathscr{H}$, then $h \in \mathscr{I} \cap \mathscr{I} = 0$. Since αf is s_α-skew, the sum only goes up to r. Now for $i \leq r$, the polynomial h_i must vanish on the zero set of α and could be written as $h_i = \alpha h_i^{'}$ for some $h_i^{'} \in \mathscr{S}$. Therefore $f = \sum_{i=1}^r h_i^{'} \sigma_i \in \mathscr{I} \cap \mathscr{H} = 0$ and $f = 0$. Hence $V = 0$ and by induction $\tilde{c}$ is injective on $\mathscr{H}$.

4 The cohomology of a compact Lie group

Consider the map $\Psi : G/T \times T \to G$ given by $\Psi(gT, t) = gtg^{-1}$. It is well-defined. The weyl group W acts on T by conjugation and on $G/$ by $w \cdot gT = gn^{-1}T$, where n is any representative of w in G. Hence W acts on $H(G/T \times T) = H(G/T) \times H(T)$. Since $\Psi(gn^{-1}T, wtw^{-1}) = \Psi(gT, t)$, the image of any form of G by ψ^* is invariant by the action of W. So ψ^* induced a map $H(G) \to [H(G/T) \times H(T)]^W$. We will show that it is a isomorphism of graded rings.

Proposition 4.1. *$\psi^* : H(G) \to [H(G/T) \times H(T)]^W$ is an isomorphism of graded rings*

Proof. First, we prove that it is an injective map. This can be shown by computing the degree of Ψ. So we need to choose the orientations of G, T and G/T. Choose orientations for $\mathfrak{m}$ and $\mathfrak{t}$ and combine them we get an orientation of $\mathfrak{g}$. Extend elements of the Lie algebra to left invariant vector fields, we get both orientations for G and T. Recall that G acts on G/T and $(A_{eT})_*$ is an isomorphism between $\mathfrak{m}$ and $T_{eT}(G/T)$. Let $(A_{eT})_*$ gives the orientation of $T_{eT}(G/T)$. Then we define the orientation of each point of G/T by the action of G, because the determinant of $Ad(T)|\mathfrak{m}$ can only be 1 this choice is well-defined. Explicitly, for any point $gT \in G/T$, we choose a representative g in G identify $\mathfrak{m}$ with the tangent space $T_{gT}(G/T)$ by

$$X \to X_{gT} = \frac{d}{ds} g(expsX)T|_{s=0},\ X \in \mathfrak{m}$$

And for any point $g \in G$, we identify $\mathfrak{g}$ with the tangent space $T_g(G)$ by

$$X \to X_g = (L_g)|_* X,\ X \in \mathfrak{g}$$

Similarly we identify $\mathfrak{t}$ with the tangent space $T_t(T)$. These identifications are consistent with the orientations defined above. We compute the derivative $(\Psi)_*|_{(gT,t)}$

$$\begin{aligned}
(\Psi)_*|_{(gT,t)}(X_{gT}, 0) &= \frac{d}{ds} g(expsX)t(exp - sX)g^{-1}|_{s=0} \\
&= \frac{d}{ds} gtg^{-1}[(expsAd(gt^{-1})X)][(exp - sAd(g)X]|_{s=0} \\
&= \frac{d}{ds} gtg^{-1}[I + sAd(g)(Ad(t^{-1} - 1)X + O(s^2)]|_{s=0} \\
&= [Ad(g)(Ad(t^{-1}) - I)X]|_{gtg^{-1}}
\end{aligned}$$

For $H \in \mathfrak{t}$,

$$\begin{aligned}
(\Psi)_*|_{(gT,t)}(0, H_t) &= \frac{d}{ds} gt(expsH)g^{-1}|_{s=0} \\
&= \frac{d}{ds} gtg^{-1}[(expsAd(g)H)]|_{s=0} \\
&= [Ad(g)H]|_{gtg^{-1}}
\end{aligned}$$

Hence under the orientation-preserving identifications, $(\Psi)_*|_{(gT,t)}$ is

$$(Ad(g))(Ad(t^{-1}) - I) \oplus (Ad(g)) : \mathfrak{m} \oplus \mathfrak{t} \to \mathfrak{g} = \mathfrak{m} \oplus \mathfrak{t}$$

G and T are connected and compact, we must have $detAd(g) = 1$. And we know that $(Ad(t^{-1}) - I)$ is a map from $\mathfrak{m}$ to $\mathfrak{m}$. Hence

$$det(\Psi)_*|_{(gT,t)} = det(Ad(t^{-1}) - I)|_{\mathfrak{m}} = det(I - Ad(t)|_{\mathfrak{m}}),$$

since $detAd(t)|_{\mathfrak{m}} = 1$.

We find a regular value for the map Ψ. Let t_0 be a generic element in T whose powers is a dense set of T. $\Psi^{-1}(t_0) = \{(gT, t) : gtg^{-1} = t_0\}$. Recall the bijective map

$$i : T/Ad(W) \to G/Ad(G)$$

Therefore if $gtg^{-1} = t_0$, then there is an element wT in Weyl group such that $wtw^{-1} = t_0$. And any $g' \in G$ such that $g'tg'^{-1} = t_0$ must be in $N(T)$ because t_0 is generic. No element in W acts on T trivially. Hence $\Psi^{-1}(t_0) = \{(wT, w^{-1}t_0w) : wT \in W\}$.

We next show that Ψ preserves the orientation at each point in $\Psi^{-1}(t_0)$. By the Chavalley's basis we can compute the determinant of $(\Psi)_*$. In each subspace $span\{X_i, X_{i+v}\}$, $Ad(w^{-1}t_0w)$ has two eigenvalues $z_i, \bar{z}_i$ such that $|z_i| = 1$. Because t_0 is generic, $w^{-1}t_0w$ is also generic and $z_i \neq 1$.

$$det(\Psi)_*|_{(wT,w^{-1}t_0w)} = \prod_i (1 - z_i)(1 - \bar{z}_i) = \prod_i 2(1 - Re(z_i)) > 0$$

This shows that Ψ preserves orientation at each point in $\Psi^{-1}(t_0)$. Hence the degree of Ψ is $|\Psi^{-1}(t_0)| = |W|$. By the following lemma, we prove that Ψ^* is injective.

Lemma 4.2. *Suppose $f : M \to N$ is a smooth map between two compact oriented manifolds of the same dimension n. If $degf \neq 0$, f^* is an injective map for $H(N)$ to $H(M)$.*

Proof. Suppose $\omega \in H^p(N)$ is a nonzero cohomology class. By Poincaré duality, there is $\alpha \in H^{n-p}(N)$ such that

$$\int_N \omega \wedge \alpha \neq 0$$

Hence

$$\int_M f^*\omega \wedge f^*\alpha = degf \cdot \int_N \omega \wedge \alpha \neq 0$$

If $f^*\omega$ is exact, since $f^*\alpha$ is closed $f^*\omega \wedge f^*\alpha$ is also exact. It is a contradiction.

Then we prove that $H(G)$ and $[H(G/T) \times H(T)]^W$ have the same dimension. Hence it is an isomorphism. Recall that for any compact Lie group, there is an invariant integration. We can construction the integration by the bi-invariant form. By lemma 2.7, for any compact connected Lie group with given orientation there is a unique bi-invariant form ω of the top degree whose integral over the Lie group is one. Hence we can define the invariant integration of a function f of G by

$$\int_G f dg \equiv \int_G f\omega$$

Let $\omega_{G/T}$ be the unique left-invariant form of top degree whose integral over G/T is one. Now $\omega_G, \omega_T, \omega_{G/T}$ are all left-invariant. By the identification, we have

$$\Psi^*\omega_G|_{gtg^{-1}} = (det(\Psi)_*|_{(gT,t)})\omega_{G/T}|_{gT} \wedge \omega_T|_t$$

Therefore we have the pull-back formula for any smooth function of G.

$$\int_G f\omega_G = \frac{1}{|W|} \int_{G/T\times T} f \circ \psi(gT, t) det(I - Ad(t)|_{\mathfrak{m}})\omega_{G/T} \wedge \omega_T$$

If f is an invariant under conjugation of G, we get Weyl integration formula:

$$\int_G f\omega_G = \frac{1}{|W|}\int_T f(t)det(I - Ad(t)|_{\mathfrak{m}})\omega_T$$

Take $f = 1$, we get

$$\int_T det(I - Ad(t)|_{\mathfrak{m}})\omega_T = |W|.$$

Let f be a function of G defined by $f(g) = det(I+Ad(g))$. By decomposing $det(I+Ad(g))$ into the sum of determinants of 2^{dimG} matrixes, we find that f is the trace of $Ad(g)$ acting on $\Lambda\mathfrak{g}$. Hence, by proposition 2.8

$$dimH(G) = dim(\Lambda\mathfrak{g})^G$$

By the orthogonal relation of characters, the number of trivial representation appearing in a representation equals the inner product of this representation's character with trivial character.

$$\begin{aligned} dimH(G) &= \int_G det(I + Ad(g))\omega_G \\ &= \frac{1}{|W|}\int_T det(I + Ad(t))det(I - Ad(t)|_{\mathfrak{m}})\omega_T \\ &= \frac{2^{dimT}}{|W|}\int_T det(I - Ad(t^2)|_{\mathfrak{m}})\omega_T \end{aligned}$$

The squaring map S on T is a 2^{dimT}-fold covering map. So the degree for squaring map is 2^{dimT} and $S^*\omega_T = 2^{dimT}\omega_T$.

$$\frac{2^{dimT}}{|W|}\int_T det(I - Ad(t^2)|_{\mathfrak{m}})\omega_T = \frac{2^{dimT}}{|W|}\int_T det(I - Ad(t)|_{\mathfrak{m}})\omega_T = 2^{dimT}$$

Hence $dimH(G) = 2^{dimT}$. On the other hand, $H(G/T)$ is the regular representation of W.

$$\begin{aligned} dim[H(G/T)\otimes H(T)]^W &= \frac{1}{|W|}\sum_w \chi_{H(G/T)}(w)\chi_{H(T)}(w) \\ &= \frac{1}{|W|}\chi_{H(G/T)}(e)\chi_{H(T)}(e) \\ &= dimH(T) = 2^{dimT} \end{aligned}$$

Hence $[H(G/T)\otimes H(T)]^W$ and $H(G)$ have the same dimension and this proposition holds.

We identify $H(G/T)$ with $\mathscr{H}_2$ and $H(T)$ with Λ. From Solomon's theorem in section 2, we get the main result

Theorem 4.3 The cohomology ring H(G) with real coefficients is a bigraded exterior algebra with generators in bi-degrees $(2m_i, 1)$, for $1 \leq i \leq l$.

Example 4.4 We give a computation of the cohomology of $U(n)$. For $G = U(n)$, $dimG = n^2$ and $dim\mathfrak{t} = n$. We can choose a basis $\{H_1, ..., H_n\}$ of $\mathfrak{t}$ and then Weyl group which is isomorphic to S^n acts on the basis by permutation. Let $\{x_1, ..., x_n\}$ be the dual basis of $\mathfrak{t}^*$. Then $\mathscr{S}^W$ is the set of symmetric polynomials in n variables. We know that any symmetric polynomials is a polynomial of elementary symmetric polynomials. So the $F_1, ..., F_n$ are the n elementary symmetric polynomials in n variables. Hence $m_i = i - 1,\ 1 \leq i \leq n$. $DimH(G) = 2^n$. There are n cohomology classes $\omega_i \in H^{2i-1}(G)$ such that $\omega_{i_1} \wedge ... \wedge \omega_{i_k}$ form a basis of $H(G)$ where $1 \leq i_1 < ... < i_k \leq n$, $0 \leq k \leq n$.

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